This week

1. Lines in $\mathbb{R}^2$
2. Section 12.5: lines and planes in space
3. Application: perspective projection
### Points and vectors

**Convention**

- As from now on we will identify **points** with **terminal points of vectors in standard position**:
  
  \[
  P = \mathbf{v}
  \]

- We will abandon the notation \( \langle x_1, \ldots, x_n \rangle \) and use \( (x_1, \ldots, x_n) \) instead.

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### Parametrisation

**Definition**

A line in \( \mathbb{R}^2 \) is defined by an equation of the form

\[
\ell: ax + by = c
\]

(*)

with \( a, b \) and \( c \) real numbers.

- The line \( \ell \) consists of the points that satisfy equation (*):

  \[
  \ell = \{(x, y) \mid ax + by = c\}.
  \]

- The line \( \ell \) is the **solution set** of equation (*).
Definition

A parametrisation of the line $\ell$ is a function $r : \mathbb{R} \to \mathbb{R}^2$ such that $r(t)$ reaches all points of $\ell$ while $t$ runs through all real numbers.

- The number $t$ is called the parameter.
- The line $\ell$ is the set of all points $r(t)$:
  \[ \ell = \{ r(t) \mid t \in \mathbb{R} \} . \]
- The function $r(t)$ has two components that both depend on $t$:
  \[ r(t) = (x(t), y(t)) . \]
- Functions like $r$ with values in $\mathbb{R}^n$ are called vector functions.

Example

Given is the line $\ell : 2x + 3y = 6$. Find a parametrisation of $\ell$.

- Choose $x$ as parameter: $t = x$.
- Solve $y$ from the equation $2t + 3y = 6$:
  \[ y = \frac{6 - 2t}{3} = 2 - \frac{2}{3}t . \]
- A parametrisation of $\ell$ is
  \[ \ell : r(t) = (t, 2 - \frac{2}{3}t) , \quad t \in \mathbb{R} . \]

\[
\begin{array}{c|ccc}
  t & x(t) & y(t) & r(t) \\
  \hline
  0 & 0 & 2 & (0, 2) \\
  1.5 & 1.5 & 1 & (1.5, 1) \\
  3 & 3 & 0 & (3, 0) \\
\end{array}
\]
### Example

Find an equation for the line

\( \ell: (3t, 2 - 2t), \ t \in \mathbb{R}. \)

- The **parametric equations** are
  \[
  \begin{aligned}
  x &= 3t, \\
  y &= 2 - 2t.
  \end{aligned}
  \]
- Eliminate \( t \): from the first parametric equation follows
  \( t = \frac{x}{3}. \)
- From the second parametric equation follows
  \[
  \begin{aligned}
  y &= 2 - \frac{2}{3}x, \\
  3y &= 6 - 2x, \\
  2x + 3y &= 6.
  \end{aligned}
  \]

### Support- and direction vector

#### Theorem

For every line \( \ell \) there exist numbers \( p_1, p_2, v_1 \) and \( v_2 \) such that

\[
\mathbf{r}(t) = (p_1 + v_1 t, p_2 + v_2 t) \quad t \in \mathbb{R}.
\]

- Write \( \mathbf{r}(t) \) as follows:
  \[
  \mathbf{r}(t) = (p_1, p_2) + t(v_1, v_2).
  \]
- The vector \( \mathbf{p} = (p_1, p_2) \) is called a **support vector** of \( \ell \).
- The vector \( \mathbf{v} = (v_1, v_2) \) is called a **direction vector** of \( \ell \).
- Define \( \mathbf{q} = \mathbf{r}(1) \), then
  \[
  \mathbf{r}(1) = \mathbf{p} + \mathbf{v}, \quad \text{dus} \quad \mathbf{v} = \mathbf{q} - \mathbf{p}.
  \]
- The **parametrised vector form** of \( \ell \) is
  \[
  \ell: \mathbf{r}(t) = \mathbf{p} + t\mathbf{v} \quad t \in \mathbb{R}.
  \]
Example

Find a support- and a direction vector of the line $\ell$: $2x + 3y = 6$, and find a parametrised vector form of $\ell$.

- A parametrisation of $\ell$ is $\ell: \mathbf{r}(t) = (t, 2 - \frac{2}{3}t), \ t \in \mathbb{R}$.
- Write $\mathbf{r}(t)$ as follows: $\mathbf{r}(t) = (0, 2) + t(1, -\frac{2}{3})$.
- Choose a support- and a direction vector $\mathbf{p} = (0, 2)$ and $\mathbf{v} = (1, -\frac{2}{3})$.

Example

Find a parametrisation and an equation of the line $\ell$ that passes through the points $P = (-1, -1)$ and $Q = (1, 3)$.

- Define $\mathbf{p} = (-1, -1)$ and $\mathbf{q} = (1, 3)$.
- Define $\mathbf{v} = \mathbf{q} - \mathbf{p} = (2, 4)$, then a parametrisation of $\ell$ is $\ell: \mathbf{r}(t) = \mathbf{p} + t\mathbf{v} = (-1, -1) + t(2, 4) = (2t - 1, 4t - 1)$.
- The parametric equations are
  \[
  \begin{align*}
  x &= 2t - 1, \\
  y &= 4t - 1,
  \end{align*}
  \]
  hence $t = \frac{x + 1}{2}$.
- Substitute this in the second equation:
  $y = 2(x + 1) - 1 = 2x + 1$ or $y - 2x = 1$.
Exercises

1. Let $P = (2, 1)$ and $Q = (-1, -1)$. The line passing through $P$ and $Q$ is called $\ell$.
   (i) Find an equation of $\ell$.
   (ii) Find a direction vector and a support vector of $\ell$.

2. The line $\ell$ is defined by the equation $y = 2x + 1$. Find a parametrised vector form for $\ell$.

3. The line $\ell$ has parametrisation $\mathbf{x}(t) = \left(1 - \frac{1}{2}t, t - 1\right)$ with $t \in \mathbb{R}$.
   (i) Find an equation of $\ell$.
   (ii) Find the intersection of $\ell$ and the line $m$ defined by the parametrised vector form $(-2, 2) + t(1, 1)$.

4. The line $\ell$ has support vector $(3, 2)$ and direction vector $(2, -1)$. The line $m$ has support vector $(-2, -3)$ and direction vector $(1, 2)$. Find the intersection point of $\ell$ and $m$.

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**Definition**

Let $\mathbf{p}$ and $\mathbf{v} \neq \mathbf{0}$ be vectors. The **parametrised vector form** of the line through $\mathbf{p}$ and parallel to $\mathbf{v}$ is

$$\mathbf{r}(t) = \mathbf{p} + t\mathbf{v}, \quad t \in \mathbb{R}.$$ 

- The vector $\mathbf{p}$ is called a **support vector** and the vector $\mathbf{v}$ is called a **direction vector** of the line.
- If $\mathbf{r}(t) = (f(t), g(t), h(t))$, then the equations
  \[
  \begin{cases}
  x = f(t), \\
  y = g(t), \\
  z = h(t)
  \end{cases}
  \]
  are called the **parametric equations** of the line.
Example

Find the parametric equations of the line \( \ell \) through \((-2, 0, 4)\) in the direction
\[
\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} = (2, 4, -2).\]

- Define \( \mathbf{p} = P_0 = (-2, 0, 4) \).
- A parametrisation of \( \ell \) is
\[
\ell : \mathbf{r}(t) = \mathbf{p} + t\mathbf{v} = (-2, 0, 4) + t(2, 4, -2) = (2t - 2, 4t, 4 - 2t).
\]
- The parametric equations of \( \ell \) are
\[
x = 2t - 2, \quad y = 4t, \quad z = 4 - 2t, \quad t \in \mathbb{R}.
\]

Example

Find the parametric equations of the line \( \ell \) through \( P = (-3, 2, -3) \) and \( Q = (1, -1, 4) \).

- Define \( \mathbf{p} = \overrightarrow{OP} = (-3, 2, -3) \) and \( \mathbf{v} = \overrightarrow{PQ} = (1, -1, 4) - (-3, 2, -3) = (4, -3, 7) \).
- A parametrisation of \( \ell \) is
\[
\ell : \mathbf{r}(t) = \mathbf{p} + t\mathbf{v} = (-3, 2, -3) + t(4, -3, 7) = (4t - 3, 2 - 3t, 7t - 3).
\]
- The parametric equations of \( \ell \) are
\[
x = 4t - 3, \quad y = 2 - 3t, \quad z = 7t - 3, \quad t \in \mathbb{R}.
\]
Summary

- A parametrisation of the line through a point \( P \) parallel to a vector \( \mathbf{v} \neq \mathbf{0} \) is
  \[ \mathbf{p} + t \mathbf{v}, \quad t \in \mathbb{R}, \]
  with support vector \( \mathbf{p} = \overrightarrow{OP} \) and direction vector \( \mathbf{v} \).
- A parametrisation of the line through two points \( P \) and \( Q \) is
  \[ \mathbf{p} + t \mathbf{v}, \quad t \in \mathbb{R} \]
  with support vector \( \mathbf{p} = \overrightarrow{OP} \) and direction vector \( \mathbf{v} = \overrightarrow{PQ} \).

Warning

Parametrisations are not unique:

- Every point on the line can be chosen as support vector.
- Every non-zero vector parallel to the line can be chosen as direction vector.

Intersection of lines in \( \mathbb{R}^3 \)

- Suppose two lines \( \ell \) and \( m \) have parametrised vector forms \( \mathbf{p} + t \mathbf{v} \) and \( \mathbf{q} + s \mathbf{w} \) respectively.
- An intersection is found if there are values for \( t \) and \( s \) such that
  \[ \mathbf{p} + t \mathbf{v} = \mathbf{q} + s \mathbf{w}. \quad (*) \]
- Since vector equations in \( \mathbb{R}^3 \) yield three equations, equation (\( * \)) may fail to have a solution, even if \( \ell \) and \( m \) are not parallel.
- Non-parallel lines that do not intersect are called skew.
Example 2.6

Let \( \ell \) be the line with support vector \((-3, -3, 1)\) and direction vector \((2, 1, 1)\). Let \( m \) be the line with support vector \((2, -3, -2)\) and direction vector \((-1, 2, 4)\). Determine if \( \ell \) and \( m \) intersect, and if so, find the intersection point.

- Solve \( s \) and \( t \) from
  \[
  -3 + 2t = 2 - s \\
  -3 + t = -3 + 2s \\
  1 + t = -2 + 4s
  \]
- From the first equation follows: \( s = 5 - 2t \).
- Substitute this in the second equation:
  \[
  -3 + t = -3 + 2(5 - 2t) = 7 - 4t
  \]
  From this follows: \( t = 2 \).
- This implies \( s = 5 - 2 \cdot 2 = 1 \).
- Now check the last equation: \( 1 + t = 3 \) and \(-2 + 4s = 2\): the equation does not hold.
- Lines \( \ell \) and \( m \) do no intersect.

Example 2.7

Let \( \ell \) be the line with support vector \((-3, -3, 0)\) and direction vector \((2, 1, 1)\). Let \( m \) be the line with support vector \((2, -3, -2)\) and direction vector \((-1, 2, 4)\). Determine if \( \ell \) and \( m \) intersect, and if so, find the intersection point.

- Solve \( s \) and \( t \) from
  \[
  -3 + 2t = 2 - s \\
  -3 + t = -3 + 2s \\
  t = -2 + 4s
  \]
- From the first equation follows: \( s = 5 - 2t \).
- Substitute this in the second equation:
  \[
  -3 + t = -3 + 2(5 - 2t) = 7 - 4t
  \]
  From this follows: \( t = 2 \).
- This implies \( s = 5 - 2 \cdot 2 = 1 \).
- Now check the last equation: \( t = 2 \) and \(-2 + 4s = 2\): the equation holds.
- The intersection point is \((1, -1, 2)\).
Planes in space

**Definition**

A plane in $\mathbb{R}^3$ is defined by an equation of the form

$$M : ax + by + cz = d$$

with $a$, $b$, $c$ and $d$ real numbers.

**Examples:**

- The plane $M_1$ defined by
  $$M_1 : x + y + z = 1$$
  passes through the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

- The plane $M_2$ defined by
  $$M_2 : x + y + z = 0$$
  passes through $O$ and is parallel to $M_1$.

- The plane $M_3$ defined by
  $$M_3 : 2y = 3$$
  is the plane through $(0, 3/2, 0)$ parallel to the $xz$-plane.
Support vectors

**Definition**

A support vector of a plane $M$ is a vector $p = \overrightarrow{OP}$ with $P$ a point of $M$.

- Suppose $M$ is defined by $ax + by + cz = d$, and let $P = (x_0, y_0, z_0)$ be a point in $M$, then $ax_0 + by_0 + cz_0 = d$, hence
  $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$
  for all $(x, y, z)$ in $M$.

**Definition**

The equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

is called the vector equation of $M$.

Normal vectors

**Definition**

A normal vector of a plane $M$ is a vector $n \neq 0$ that is perpendicular to $M$.

- Let $M$ be a plane defined by the vector equation
  $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$
  then
  $$\langle a, b, c \rangle \cdot (x - x_0, y - y_0, z - z_0)$$
  $$= \langle a, b, c \rangle \cdot ((x, y, z) - (x_0, y_0, z_0))$$
  $$= 0$$
  for all $(x, y, z)$ in $M$.

- Define $x = (x, y, z)$, $p = (x_0, y_0, z_0)$ and $n = (a, b, c)$, then
  $$n \cdot (x - p) = 0$$
  for all $x$ in $M$.

**Definition**

The equation $n \cdot (x - p) = 0$ is called the normal equation of $M$.
The normal equation

**Theorem**

Let $M$ be defined by the normal equation $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$, where $\mathbf{n}$ is a normal vector of $M$, and let $\mathbf{p} = (x_0, y_0, z_0)$ be a support vector. If $X = (x, y, z)$ is a point of $M$ then $\mathbf{n} \perp \overrightarrow{PX}$.

- Note that $\overrightarrow{PX} = \mathbf{x} - \mathbf{p}$.

The normal equation

**Example**

*Find an equation of the plane $M$ through $(-3, 0, 7)$ orthogonal to $\mathbf{n} = (5, 2, -1)$.*

- Define $\mathbf{p} = (-3, 0, 7)$, then the normal equation $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$ gives:
  $$ (5, 2, -1) \cdot ((x, y, z) - (-3, 0, 7)) = 0, $$

  or
  $$ (5, 2, -1) \cdot (x + 3, y, z - 7) = 0. $$

- The vector equation of $M$ is
  $$ 5(x + 3) + 2y - (z - 7) = 0. $$

- Simplification gives
  $$ 5x + 2y - z = -22. $$
The normal equation

Example

*Find a normal equation for the plane $M: y - 2z = 4$.*

- Write the equation as follows:
  $$0 \cdot x + 1 \cdot y + (-2) \cdot z = 4.$$
- A normal vector is $\mathbf{n} = (0, 1, -2)$.
- The components of $\mathbf{n}$ are the coefficients of the equation.
- For a point $P$ in the plane we choose $x = z = 0$. Then $y = 4$, so $P = (0, 4, 0)$ gives support vector $\overrightarrow{OP} = \mathbf{p} = (0, 4, 0)$.
- A normal equation of $M$ is
  $$(0, 1, -2) \cdot (x - (0, 4, 0)) = 0.$$
- *Every* point of $M$ can be used as support vector, for instance $\mathbf{p}' = (1, 6, 1)$ also works.

A plane through three points

Example

*Find an equation for the plane $M$ through the points $A = (0, 0, 1)$, $B = (2, 0, 0)$ and $C = (0, 3, 0)$.*

- Suppose the equation is $ax + by + cz = d$ with yet to determine constants $a$, $b$, $c$ and $d$.
- The points $A$, $B$ and $C$ all lie in the plane, this gives three equations:
  $$A \in M \Rightarrow c = d \Rightarrow c = d$$
  $$B \in M \Rightarrow 2a = d \Rightarrow a = \frac{1}{2}d$$
  $$C \in M \Rightarrow 3b = d \Rightarrow b = \frac{1}{3}d$$
- Substitute this in the equation: $\frac{1}{2}dx + \frac{1}{3}dy + dz = d$.
- Divide left- and right-hand side by $d$: $\frac{1}{2}x + \frac{1}{3}y + z = 1$.
- Avoid fractions by multiplying with 6: $3x + 2y + 6z = 6$.
Lines as intersection of two planes

- Unlike lines in $\mathbb{R}^2$, lines in $\mathbb{R}^3$ cannot be described by one equation: a linear equation $ax + by + cz = d$ describes a plane.
- In order to describe a line you need two equations:
  \[
  \begin{align*}
  ax + by + cz &= d \\
  px + qy + rz &= s 
  \end{align*}
  \]
- Regard a line as the intersection of two planes:
Example

Give a parametrisation of the line described by the equations

\[
\begin{align*}
  x + y - 2z &= -1 \\
 2x - y + z &= 2
\end{align*}
\]

- Choose one of the variables as parameter, for instance: \( x = t \)
- Replace \( x \) by \( t \) in the given equations:
  
  \[
  \begin{align*}
    y - 2z &= -1 - t \\
    -y + z &= 2 - 2t
  \end{align*}
  \]

- Express \( y \) and \( z \) in \( t \) by solving system (1). For instance, from the first equation follows
  
  \[
  y = 2z - 1 - t.
  \]
- Plug this in the second equation of (1) and solve \( z \):
  
  \[
  -(2z - 1 - t) + z = 2 - 2t \implies z = -1 + 3t
  \]

Example (continued)

- Use equation (2) to express \( y \) in \( t \):
  
  \[
  y = 2z - 1 - t = 2(-1 + 3t) - 1 - t = -3 + 5t
  \]
- Summary: the boxed equations express \( x, y \) and \( z \) in \( t \) and therefore can be used as parametric equations for the line:
  
  \[
  \begin{align*}
    x &= t \\
    y &= -3 + 5t \\
    z &= -1 + 3t
  \end{align*}
  \]
- The parametrised vector form then is
  
  \[
  \mathbf{x} = (x, y, z) = (0, -3, -1) + t(1, 5, 3).
  \]
- A support vector then is \((0, -3, -1)\), and as direction vector you can use \((1, 5, 3)\).
Check your answer!

- Check that
  \[
  \begin{align*}
  x + y - 2z &= -1 \\
  2x - y + z &= 2
  \end{align*}
  \]
  is the line through \( p = (0, \ -3, 1) \) and in direction \( v = (1, 5, 3) \).

- Let \( x = 0, \ y = -3, \ z = -1 \), then
  \[
  \begin{align*}
  x + y - 2z &= 0 - 3 + 2 = -1 \\
  2x - y + z &= 0 + 3 - 1 = 2
  \end{align*}
  \]

- The normal vectors of the planes \( x + y - 2z = -1 \) and \( 2x - y + z = 2 \) are \( \mathbf{n}_1 = (1, 1, -2) \) and \( \mathbf{n}_2 = (2, -1, 1) \) respectively.

- Check that \( v \perp \mathbf{n}_1 \) and \( v \perp \mathbf{n}_2 \):
  \[
  \mathbf{v} \cdot \mathbf{n}_1 = 1 + 5 - 6 = 0,
  \]
  and
  \[
  \mathbf{v} \cdot \mathbf{n}_2 = 2 - 5 + 3 = 0.
  \]

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Intersection of a line and a plane

**Example**

*The line \( \ell \) is defined by the parametrisation*

\[
\begin{align*}
  x &= \frac{8}{3} + 2t, \\
  y &= -2t, \\
  z &= 1 + t, \\
  t &\in \mathbb{R}.
\end{align*}
\]

*Find the intersection of \( \ell \) and the plane \( 3x + 2y + 6z = 6 \).*

- Suppose the intersection is
  \[
  \mathbf{x}_0 = \left( \frac{8}{3} + 2t, -2t, 1 + t \right). \tag{1}
  \]

- The point \( \mathbf{x}_0 \) lies on the plane, so
  \[
  3\left( \frac{8}{3} + 2t \right) + 2(-2t) + 6(1 + t) = 6.
  \]

- Solve \( t \) from this equation:
  \[
  8 + 6t - 4t + 6 + 6t = 6,
  \]
  which implies \( t = -1 \).

- The intersection is obtained by substituting \( t = -1 \) in (1):
  \[
  \mathbf{x}_0 = \left( \frac{2}{3}, 2, 0 \right).
  \]
Line through a point perpendicular to a plane

- If a plane $M$ is defined by the equation $ax + by + cz = d$, then $n = (a, b, c)$ is a normal of $M$. This is also a direction vector of the line.
- If the line passes through $q$ then the line can be parametrised by

\[ \ell: q + tn. \]

- The projection of $q$ on $M$ is $\hat{q}$, the intersection of the line $\ell$ with $M$.

Example

Let $M$ be defined by $2x - y - z = 3$, and let $q = (-2, 2, 3)$. Find the projection of $q$ on $M$.

- The coefficients of the equation for $M$ give the normal:
  \[ n = (2, -1, -1). \]
- The line through $p$ perpendicular to $M$ is parametrized by
  \[ q + tn = (-2, 2, 3) + t(2, -1, -1) = (2t - 2, -t + 2, -t + 3). \]
- The parametric equations are
  \[ x = 2t - 2, \quad y = -t + 2 \quad \text{and} \quad z = -t + 3. \]
- Substitution in the equation for $M$ gives the equation
  \[ 2(2t - 2) - (-t + 2) - (-t + 3) = 3 \quad \Rightarrow \quad 6t - 9 = 3. \]
- Solving this equation yields $t = 2$.
- Substitution of $t = 2$ in the parametric equations gives the intersection:
  \[ \hat{q} = (2, 0, 1) \]
**Theorem**

If $P$ is a point on the plane $M$, and $n$ is a normal of $M$, the distance of an arbitrary point $Q$ to $M$ is

$$d = \frac{|\overrightarrow{PQ} \cdot n|}{|n|}$$

- The distance can be found by calculating $|\mathbf{q} - \hat{\mathbf{q}}|$, where $\mathbf{q} = \overrightarrow{OQ}$ and $\hat{\mathbf{q}}$ is the projection of $\mathbf{q}$ on $M$.

**Example**

The plane $M$ is defined by $3x + 2y + 6z = 6$. Find the distance of $Q = (1,1,3)$ to $M$.

- The coefficients of the equation for $M$ give the normal: $n = (3, 2, 6)$.
- For a point of $M$, choose two values for say $x$ and $z$, then $y$ follows from the equation:
  
  $x = 0, \ z = 0 \ \Rightarrow \ y = 3$,

  hence $P = (0, 3, 0)$ is a point of $M$.
- The distance of $Q$ to $M$ is
  
  $$d = \frac{|\overrightarrow{PQ} \cdot n|}{|n|} = \frac{|((1,1,3) - (0,3,0)) \cdot (3,2,6)|}{\sqrt{3^2 + 2^2 + 6^2}} = \frac{17}{7}.$$
Parametrise the line ℓ as follows:

\[ \ell : \mathbf{r}(t) = (x_0, 0, 0) + t(x_1 - x_0, y_1, z_1), \quad t \in \mathbb{R}. \]

The intersection of ℓ and the yz-plane is \( P = \mathbf{r}(t_0) \) with \( t_0 = \frac{x_0}{x_0 - x_1} \).

For \( P = (0, y, z) \) we have

\[ y = t_0 y_1 = \frac{x_0 y_1}{x_0 - x_1} \quad \text{and} \quad z = t_0 z_1 = \frac{x_0 z_1}{x_0 - x_1}. \]